Thermal Time Constants and Dynamic Compressibility of Air in Fiber-Filled Loudspeaker Enclosures*

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In a loudspeaker enclosure filled with fibrous damping material, the conduction of heat between the fibers and the air involves a time constant whose value depends on what conditions are held constant during the heat transfer. Five combinations of conditions are considered, leading to five different time constants. One of these, denoted by τ_{fp}, is the time constant at constant fiber temperature and constant pressure. Equations expressing the other four time constants in terms of τ_{fp} are derived using a thermal circuit model. Another thermal circuit yields an approximate analytical expression for τ_{fp} in terms of the diameter and packing density of the fibers and the thermal diffusivity of air. A more accurate numerical computation of τ_{fp}, based on separation of variables, is also obtained. By adjusting a single parameter in the analytical approximation, the approximation is made to agree with the numerical solution within a tolerance of $\pm 2\%$ for all possible input data. Assuming an intuitive form for the equivalent circuit of the acoustic compliance of the enclosure, it is shown that two of the five time constants can be read from the acoustic circuit, and hence that the components in the acoustic circuit can be calculated simply and accurately from the known properties of the enclosure and the filling.

0 INTRODUCTION

The frequency response of a loudspeaker in a fiber-filled enclosure is influenced by heat conduction between the fibers and the air. When the air is compressed, its temperature and pressure initially increase. The resulting temperature differential causes a transfer of heat from the air to the fibers, which offsets some of the increase in air temperature and pressure. When the air is rarefied, the reverse process occurs. At sufficiently high audio frequencies, the heat transferred during one half-cycle is negligible compared with the work done, so that the compression of the air is nearly adiabatic, as if the fibers were absent. At sufficiently low frequencies, the fibers and the air remain nearly in thermal equilibrium, so that the compression is more nearly isothermal. Thus the fibrous filling can increase the compliance of the enclosed air [1, pp. 587–588]. At intermediate frequencies, the time scale of heat conduction is comparable with the period of the audio-frequency oscillation, so that the heat transferred to the fibers during compression is not fully restored to the air during expansion, and some of the work of compression is irreversibly converted into heat. Thus the heat conduction contributes to the damping of the enclosure at these frequencies.

The problem is to determine which frequencies are "sufficiently high" and "sufficiently low." Chase [2] and Leach [1] have considered a typical case in which the fibers are 5 μm in radius and occupy 0.25% of the volume. Assuming for simplicity that the heat transfer takes place at constant fiber temperature and constant air pressure, both authors conclude that the heat transfer is well approximated as an exponential function of time with a single time constant. If we call the time constant τ_{fp}, where the f stands for constant fiber temperature and the p for constant pressure, the two authors agree that the transition from thermal equilibrium to adiabatic behavior takes place around a frequency $f_c = (2\pi\tau_{fp})^{-1}$. But on the value of f_c for the "typical" example, the authors disagree by more than three orders of magnitude, Chase obtaining 3.5 Hz and Leach obtaining 6.4 kHz.

Hence Chase concludes that the fibers must be compacted if we wish to obtain nearly isothermal compression of the air at the lowest operating frequencies, and that uncompacted fibers "would have no effect in the audio range." Leach concludes that the air may be assumed to be in thermal equilibrium with the fibers at all

* Manuscript received 1997 February 15; revised 1997 October 9.

operating frequencies of the bass driver, and notes that his result contradicts that of Chase.

My purpose in this paper is not merely to adjudicate between Chase and Leach (I disagree with both), but rather to obtain a more general result which permits instant calculation of the thermal time constant for any diameter and packing density of the fibers. This paper also builds on one of the notable achievements of Leach, namely, his equivalent-circuit model of the compliance of a fiber-filled loudspeaker enclosure [1, p. 589]. Leach acknowledges that τ_{fp} (which he calls τ_1) is not the appropriate time constant to use in calculating the equivalent-circuit elements, but only an approximation thereto. This paper begins by examining the various ways of defining the thermal time constant, and finds an expression for Leach's desired time constant in terms of τ_{fp}.

Both Leach and Chase use separation of variables to solve the heat equation in cylindrical coordinates. The same procedure is followed here. But first an approximate analytical formula for τ_{fp} is found by a simpler and less error-prone method. The formula assumes that τ_{fp} is just an RC time constant, where R is the thermal resistance between one fiber and its share of the surrounding air, and C is the heat capacity of the same air. Further approximations are made in estimating R and C. The resulting formula can obviously be used to check the results of the separation of variables—any gross discrepancy would be evidence of theoretical or computational error. It turns out, however, that the approximate formula can be made accurate enough for the practical calculation of time constants: by adjusting a single undetermined parameter in the formula, the time constants calculated from the formula and from the separation of variables can be made to agree to within 2% for all possible filling factors. The more elaborate method, having served its purpose of refining the analytical approximation, can then be put aside.

For the typical example given by Chase and Leach, the refined analytical formula gives a transition frequency of about 289 Hz. This implies that neither the adiabatic assumption nor the equilibrium assumption is valid for all audio frequencies, and that a realistic model of the enclosure must take the thermal time constant into account.

1 LIST OF SYMBOLS

a	Radius of fibers
C	Heat capacity
C_a, C_{th}, R_{th}	Acoustic circuit components
C_f	Specific heat of fibers
C_p	Specific heat of air at constant pressure
C_v	Specific heat of air at constant volume
d	Diameter of fibers, $= 2a$
f	Filling factor (packing factor)
G_{th}	Thermal conductance per unit volume
m	$= f^{-1/2}$
p	Excess pressure (pressure rise above ambient)
P_0	Ambient atmospheric pressure
q	Heat transfer per unit volume
$\mathbf{q}$	Heat flux density vector (power per area)
r, ϕ, z	Cylindrical coordinates
R	Thermal resistance
S	Space-dependent factor (separation of variables)
t	Time
T	Time-dependent factor (separation of variables) or temperature
T_0	Ambient temperature
V	Volume
x	Normalized radius, $= r/a$
y	Radial eigenfunction, $y(x) = S(r)$
α	Thermal diffusivity
β	$f \rho_f C_f/[(1 - f)\rho_0 C_v]$
γ	$= C_p/C_v$
ζ	Undetermined index in formula for τ_{fp}
θ	Excess temperature of air
θ_f	Excess temperature of fibers
κ	Thermal conductivity
μ_n	nth Sturm–Liouville eigenvalue
ρ_0	Mean density of air
ρ_f	Intrinsic (bulk) density of fibers
τ_n	Time constant corresponding to μ_n
τ_{fp}	Thermal time constant at constant fiber temperature and pressure
τ_{fv}	Thermal time constant at constant fiber temperature and volume
τ_p	Thermal time constant with no external heat flow and constant pressure
τ_v	Thermal time constant with no external heat flow and constant volume
τ_a	Thermal time constant at constant air temperature

2 THE THERMAL CIRCUIT

In a volume element inside a fiber-filled loudspeaker enclosure, let the fraction of the volume occupied by fibers (the filling factor or packing factor) be f. Let ρ_0 be the mean density of the air (mass per unit volume of air) and ρ_f the bulk density of the fibers (mass per unit volume of the material of which the fibers are made). These densities are intrinsic; the corresponding average densities (mass per unit *overall* volume) are $(1 - f)\rho_0$ for the air and $f\rho_f$ for the fibers. The specific heats mentioned in Section 1 are mass-specific heats, that is, heat capacities per unit mass. To convert these to heat capacities per unit overall volume, we multiply by the corresponding average densities, obtaining $(1 - f)\rho_0 C_p$ for the air at constant pressure, $(1 - f)\rho_0 C_v$ for the air at constant volume, and $f\rho_f C_f$ for the fibers.

Let the excess temperature (temperature rise above ambient) be θ for the air and θ_f for the fibers, and let θ and θ_f be understood as spatial averages. Suppose that, starting from thermal equilibrium, we somehow transfer heat from the fiber to the air. Suppose further that the transfer takes place at constant pressure and with no external heat flow, that is, with no net heat flow into or out of the air–fiber system. Let q denote the transferred energy per unit overall volume. Then for the air,

$$q = (1 - f)\rho_0 C_p \theta \tag{1}$$

and for the fibers,

$$-q = f \rho_f C_f \theta_f . \tag{2}$$

If heat energy per unit volume is represented by charge, and excess temperature by voltage, Eqs. (1) and (2) show that the volume-specific heats are analogous to capacitances. Now let G_{th} denote the thermal conductance per unit overall volume between the air and the fiber (assuming that the rate of heat transfer is proportional to the temperature difference and to the air–glass surface area, the latter being proportional to overall volume). Then

$$\dot{q} = G_{th}(\theta_f - \theta) \tag{3}$$

so that G_{th} is analogous to electrical conductance.

Eqs. (1)–(3) are modeled by the thermal circuit shown in Fig. 1. Ground potential represents the ambient temperature T_0. Nodal temperatures are relative to T_0 and are written "in the nodes." The conductance G_{th} is shown as a resistance $1/G_{th}$.

3 FIVE THERMAL TIME CONSTANTS

The time constant of the series circuit in Fig. 1 is the thermal time constant between the air and the fiber at constant pressure, with no external heat flow. If we call this time constant τ_p, where the subscript p stands for constant pressure, the equivalent circuit gives

$$\tau_p = \frac{1}{G_{th}} [(1 - f)\rho_0 C_p \| f \rho_f C_f] \tag{4}$$

where $\|$ is the harmonic sum operator, defined by

$$u \| v \triangleq \frac{1}{1/u + 1/v} \equiv \frac{uv}{u + v} \tag{5}$$

and having a precedence below multiplication and division, but above addition and subtraction. From the definition it is easily shown that multiplication is distributive over harmonic addition, that is, for all u, v, k,

$$k(u \| v) = ku \| kv . \tag{6}$$

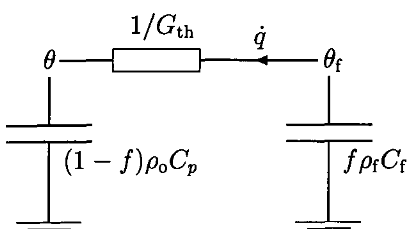

Fig. 1. Air–fiber thermal circuit at constant pressure.

Recall that τ_{fp} is the thermal time constant for constant fiber temperature and constant pressure (the maintenance of constant fiber temperature would require some external heat flow). To impose constant θ_f, we can either "short out" $f \rho_f C_f$ in Fig. 1 or let $C_f \rightarrow \infty$ in Eq. (4). In either case, the result is

$$\tau_{fp} = \frac{1}{G_{th}} (1 - f)\rho_0 C_p . \tag{7}$$

Now let τ_v and τ_{fv} be defined like τ_p and τ_{fp}, except that the subscript v (instead of p) indicates constant volume (instead of constant pressure). At constant volume, the thermal circuit is the same as Fig. 1, except that C_v replaces C_p. Making the same replacement in Eqs. (4) and (7) gives

$$\tau_v = \frac{1}{G_{th}} [(1 - f)\rho_0 C_v \| f \rho_f C_f] \tag{8}$$

$$\tau_{fv} = \frac{1}{G_{th}} (1 - f)\rho_0 C_v . \tag{9}$$

For the sake of completeness, let τ_a be the thermal time constant at constant air temperature, that is, at constant θ. Then τ_a can be found by shorting out the heat capacity of the air in Fig. 1, or letting $C_p \rightarrow \infty$ in Eq. (4), or letting $C_v \rightarrow \infty$ in Eq. (8). The result is

$$\tau_a = \frac{1}{G_{th}} f \rho_f C_f . \tag{10}$$

Now recall that

$$C_p = \gamma C_v \tag{11}$$

and let

$$\beta = \frac{f \rho_f C_f}{(1 - f)\rho_0 C_v} . \tag{12}$$

Notice that β, like γ, is a ratio of specific heats: the numerator is the heat capacity of the fibers per unit overall volume, and the denominator is the heat capacity of the air per unit overall volume. Using Eqs. (11) and (12) and the distributive law [Eq. (6)], we can easily express all the time constants in terms of τ_{fp}. Dividing Eq. (4) by Eq. (7) yields

$$\tau_p = \frac{\beta \tau_{fp}}{\gamma + \beta} . \tag{13}$$

Similarly, from Eqs. (8) and (7),

$$\tau_v = \frac{\beta \tau_{fp}}{\gamma(1 + \beta)} \tag{14}$$

and from Eqs. (9) and (7),

$$\tau_{fv} = \frac{\tau_{fp}}{\gamma} \tag{15}$$

and from Eqs. (10) and (7),

$$\tau_a = \frac{\beta\tau_{fp}}{\gamma}. \qquad (16)$$

Leach [1, p. 592] uses τ_{fp} (which he calls τ_1) as an estimate of τ_p (which he calls τ_f) and notes that $\tau_p < \tau_{fp}$. This is confirmed by Eq. (13).

4 ASSUMPTIONS

Section 3 treated the air and fiber temperatures as spatial averages and assumed nothing concerning the temperature distributions within the air and the fiber. This approach was sufficient for determining the relationships between the time constants. But the calculation of any one time constant must take these distributions into account.

First it will be shown that, at least in the case of glass fibers, the temperature within each fiber may be assumed uniform. According to Fourier's law of heat conduction [3, p. 4-143],

$$\boldsymbol{q} = -\kappa\nabla T \qquad (17)$$

where $\boldsymbol{q}$ is the heat flux density (power per unit area), κ is the thermal conductivity, and T is the temperature (absolute, or relative to an arbitrary reference). At the air–glass surface, heat is transferred between the air and the glass by the normal component of $\boldsymbol{q}$. By conservation of energy, the normal component of $\boldsymbol{q}$ is continuous across the surface. Hence, by Eq. (17), a step change in κ must be accompanied by a reciprocal step change in the normal component of ∇T, that is, the ∇T ratio is the reciprocal of the κ ratio. The thermal conductivity of air at room temperature is about 0.026 $\mathrm{Wm^{-1}K^{-1}}$ [4, p. 962]. Common soda-lime glasses have thermal conductivities around 1 $\mathrm{Wm^{-1}K^{-1}}$ [5, pp. 12-143–146]; conductivities of other common glasses are of the same order of magnitude. From the figures cited, the thermal conductivity of the glass is roughly 40 times that of the air, so that the temperature gradient on the glass side of the surface is roughly 1/40 of that on the air side. Hence it is reasonable to neglect the spatial variation of temperature within the glass.

If we assume uniform temperature within the fibers, the "constant" fiber temperature assumed in the definitions of τ_{fp} and τ_{fv} becomes a constant *and* uniform fiber temperature. This suggests that τ_{fp} and τ_{fv} are easier to calculate than the other time constants, because they involve a simple constant-temperature boundary condition on the surface of the fiber.

Next we must consider the pattern of heat conduction around each fiber. Suppose the air in some local region is compressed, so that the air temperature rises above the fiber temperature and heat begins to flow from the air to the fibers. Over the surface of each fiber, the heat flux is inward. Hence, between any two fibers, there must exist a surface across which the normal component of the heat flux density is zero; heat flows away from this surface on both sides. Let us call this surface the "heatshed" (by analogy with watershed). In the presence of many fibers, the heatshed is a honeycomb-like surface which divides the space between the fibers into a myriad of contiguous tubes, which we shall call "heat tubes." Each tube is threaded by one fiber. (More generally, a heatshed is a surface across which there is no heat flux. If the air is cooler than the fiber, the flow of heat on both sides of the heatshed is toward it instead of away from it—a slight loosening of the analogy between heatshed and watershed.)

In any heat tube, the maximum heat flux density, hence the maximum temperature gradient, occurs at the surface of the fiber. The temperature gradient is small in the outer regions of the heat tube, reaching zero at the outer surface. Hence the position and the shape of the outer surface have little effect on the effective thermal resistance between the fiber and the air in the heat tube. Moreover, the position and the shape of the heatshed do not affect the average heat capacity of the air in a heat tube. A shift in the heatshed causes an increase in the heat capacity of one heat tube and a compensating decrease in that of another. Therefore, for the purpose of calculating the thermal time constant, we may approximate the heatshed surrounding each heat tube by a surface of "average shape" containing the correct average volume; it does not greatly matter that individual heat tubes have different shapes or volumes, or even that the heatshed may move with time.

The geometry of the fibers is more critical because of the high temperature gradients at the fiber surfaces. To obtain a well-defined and realistic problem, let us suppose that the fibers are of circular cross section and uniform diameter. In practice this means that the fibers are of a synthetic material such as glass. (We may hope that the thermal behavior of natural fibers, with their variations in thickness and geometry, can be modeled satisfactorily by assuming cylindrical fibers of a suitably defined average diameter, but this question is not examined in the present paper.)

In a sufficiently small region, each fiber may be assumed cylindrical, that is, the curvature of the axis of each fiber may be neglected. Hence it is most convenient to assume that the surrounding heatshed is a cylindrical surface coaxial with the fiber. Let the radii of the fiber and the heatshed be a and ma, respectively, where $m > 1$. Then the filling factor is

$$f = \frac{\pi a^2}{\pi (ma)^2} = m^{-2} \qquad (18)$$

so that

$$m = f^{-1/2}. \qquad (19)$$

Consider a segment of fiber of length l. Let us adopt a cylindrical coordinate system coaxial with the fiber and heatshed, with the origin at one end of the fiber segment. If r is the radial coordinate, the air occupies the region

$$a \leqslant r \leqslant ma, \qquad 0 \leqslant z \leqslant l \qquad (20)$$

and the glass occupies the region

$$r \leqslant a, \qquad 0 \leqslant z \leqslant l \tag{21}$$

(see Fig. 2).

5 ANALYTICAL APPROXIMATION

If thermal resistance is analogous to electrical resistance, then thermal conductivity κ is analogous to electrical conductivity σ, so that the thermal resistance of the cylindrical shell between radius r and radius $r + \mathrm{d}r$ is

$$\mathrm{d}R = \frac{\mathrm{d}r}{\kappa 2\pi r l} . \tag{22}$$

To estimate the total thermal resistance between the fiber and the air, let us consider the heat capacity of the air to be concentrated at the "representative radius"

$$r = \frac{m+1}{2} a \tag{23}$$

that is, midway between the fiber surface and the heatshed. Then the total thermal resistance is

$$R = \int_a^{(m+1)a/2} \frac{\mathrm{d}r}{\kappa 2\pi r l} = \frac{1}{2\pi l \kappa} \ln\left(\frac{m+1}{2}\right) . \tag{24}$$

Notice that the representative radius determines the argument of the ln function, which is a weak function for large arguments. So for large m, that is, for small filling factors, the choice of the representative radius does not greatly affect the accuracy of the result. Also notice that the expression for R is defined and positive for $m > 1$, that is, for all meaningful values of m. Unnecessary limits on the range of m have been avoided so that we will not be prevented from exploiting any knowledge of the asymptotic behavior of τ_{fp} as m approaches unity or infinity.

Concerning the heat capacity of the air in the heat tube, the first question is whether we should use C_v or C_p as the specific heat. It is tempting to say simply that we must use C_p to find τ_{fp}, and C_v to find τ_{fv}. But the accurate solution of the heat equation allows different regions of air to be heated or cooled at different rates. Therefore, for consistency in the derivation, we must seek a condition common to all regions. If we use C_v, we assume that heat is conducted so fast that local regions of air do not have time to expand or contract as they are heated or cooled. This would require the conducted heat to propagate much faster than sound. If we use C_p, we assume that heat conduction is so slow that any local variations in pressure with temperature have ample time to dissipate or "equalize"—in other words, that acoustic disturbances propagate much faster than thermal disturbances, in accordance with the assumption of adiabatic compression in acoustics. So we can *consistently* use C_p, but not C_v; hence we can directly calculate τ_{fp}, but not τ_{fv}.

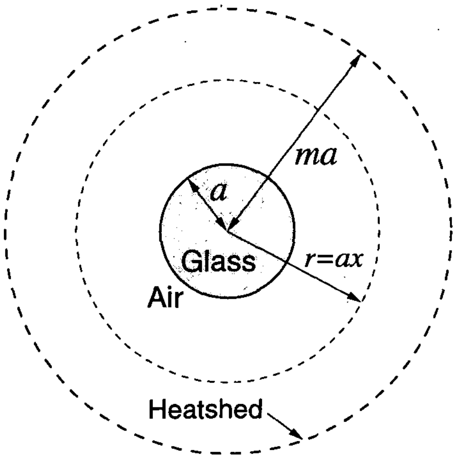

Fig. 2. Cross section of average heat tube. Outer surface (radius ma) is heatshed. Air region is described by $a < r < ma$ or $1 < x < m$.

For constant pressure, the heat capacity of the air is C_p multiplied by the effective mass of air, that is,

$$C = \rho_0 C_p A l \tag{25}$$

where A is the effective cross-sectional area. We could assume that A is simply the total air area, that is, the annular area between the fiber and the heatshed. However, to allow for the fact that the temperature variations close to the fiber (modeled as an infinite heat sink) must be less that those further out, it is desirable to exclude an area somewhat greater than that of the cross section of the fiber. So let us assume

$$A = \pi(ma)^2 - \epsilon\pi a^2 = (m^2 - \epsilon)\, \pi a^2 \tag{26}$$

where

$$1 \leqslant \epsilon < m^2 . \tag{27}$$

Notice that A becomes the total air area if we take $\epsilon = 1$.

Substituting Eq. (26) into Eq. (25), then multiplying our expressions for R and C, we obtain the time constant

$$\tau_{fp} = \frac{a^2}{2\alpha} (m^2 - \epsilon) \ln\left(\frac{m+1}{2}\right) \tag{28}$$

where α, known as the thermal diffusivity, is defined as [3, p. 4-144]

$$\alpha = \frac{\kappa}{\rho_0 C_p} . \tag{29}$$

The remaining problem is to find ϵ. This is where it is useful to consider the asymptotic behavior for large and small values of m. Because the greatest temperature gradients occur near the fiber, we should exclude only a small fraction of the cross section for large m, that is, $\epsilon << m^2$ for large m. If m is close to unity, the temperature gradient will be significant throughout the annular cross section of the air, so that we need to exclude a substantial fraction of this annular area by making ϵ

significantly greatly than unity. We can satisfy both the large-m and small-m requirements, as well as the fundamental constraint (27), by taking

$$\epsilon = m^{\zeta}, \quad 0 \leq \zeta < 2 . \tag{30}$$

Substituting this into Eq. (28), we obtain

$$\tau_{fp} = \frac{a^2}{2\alpha}(m^2 - m^{\zeta}) \ln\left(\frac{m+1}{2}\right) \tag{31}$$

or, in terms of the fiber diameter d,

$$\tau_{fp} = \frac{d^2}{8\alpha}(m^2 - m^{\zeta}) \ln\left(\frac{m+1}{2}\right) . \tag{32}$$

Part of the motivation for Eq. (30) was that any value of ζ less than 2 gives an accurate cross section and heat capacity for *large* values of m. We have also noted that any reasonable representative radius in Eq. (24) gives an accurate thermal resistance for large m. Hence, for large m, Eq. (31) is quite accurate for all permissible values of ζ. Therefore we choose ζ by considering *small* values of m. Because $m > 1$, a small value of m is

$$m = 1 + \delta \tag{33}$$

where $0 < \delta << 1$. Hence we have

$$m^2 \approx 1 + 2\delta \tag{34}$$

$$m^{\zeta} \approx 1 + \zeta\delta \tag{35}$$

$$\ln\left(\frac{m+1}{2}\right) \approx \frac{\delta}{2} . \tag{36}$$

Substituting the small-m approximations into Eq. (31) gives

$$\tau_{fp} \approx \frac{a^2}{2\alpha} \frac{2-\zeta}{2} \delta^2 \tag{37}$$

so that $\tau_{fp} \propto \delta^2$, if and only if ζ is constant. Let us check this proportionality. If δ is very small, the air cross section is a thin annulus whose area is nearly proportional to its width, which in turn is proportional to δ. Hence the heat capacity and the thermal resistance of the air are both proportional to δ, and their product is proportional to δ^2. So we can indeed take the index ζ to be a constant, and an appropriate choice of ζ will make Eq. (31) agree closely with the full solution of the heat equation for small m. We may then hope that the adjusted analytical formula for τ_{fp} makes a smooth transition from large-m to small-m behavior, giving acceptable accuracy for all m.

(*Note*: While Eq. (32) may accurately predict the thermal time constant of a *coaxial cylindrical* heat tube for all m, the assumption that the heat tube is coaxial and cylindrical is valid only for sufficiently large m, that is, for sufficiently small packing factors. With densely packed fibers we can no longer say that the temperature gradient is small in regions far from the fibers and that such regions make negligible contribution to the overall thermal resistance, because there are no such regions.)

6 SOLVING THE HEAT EQUATION

The temperature field in the air is described by the heat equation [3, p. 4-144]

$$\dot{\theta} = \alpha \nabla^2 \theta \tag{38}$$

where θ is the excess temperature (temperature rise above ambient) and α is the thermal diffusivity defined by Eq. (29). For our assumed geometry we may use cylindrical coordinates and write $\theta = \theta(r, t)$. By symmetry, θ is independent of the other coordinates ϕ and z. If the fiber is held at the ambient temperature, its excess temperature is zero, giving the boundary condition

$$\theta(a, t) = 0 \tag{39}$$

for the inner boundary. The outer boundary is the heatshed. Absence of heat flow across the heatshed implies the zero-temperature-gradient boundary condition

$$\frac{\partial \theta}{\partial r}(ma, t) = 0 . \tag{40}$$

For the initial condition, we shall follow the existing literature by supposing that we have thermal equilibrium for $t < 0$ and that the air is subject to a step increase in pressure at time $t = 0$, causing a uniform step increase in temperature. If the initial temperature rise is θ_0, we have the initial condition

$$\theta(r, 0) = \theta_0, \qquad a < r < ma . \tag{41}$$

The solution to Eqs. (38)–(41) will be presented in sufficient detail to show the points of disagreement with Chase [2] and Leach [1].

6.1 Separation of Variables

Suppose Eq. (38) has the product solution

$$\theta = S(r)T(t) . \tag{42}$$

Substituting this into Eq. (38) and dividing through by ST gives the separated form

$$\frac{\dot{T}}{T} = \alpha \frac{\nabla^2 S}{S} . \tag{43}$$

The left-hand side shows that the separation constant has the dimensions of $(\text{time})^{-1}$; we may therefore let the constant be $-1/\tau$, where τ is a time. Setting both sides equal to this constant gives

$$\dot{T} = -\frac{T}{\tau} \tag{44}$$

$$\nabla^2 S + \frac{1}{\alpha\tau} S = 0 . \qquad (45)$$

Eq. (44) has the solution

$$T = A e^{-t/\tau} \qquad (46)$$

where A is a constant, showing that τ is a time constant in the usual sense and that τ must be positive if the solution is to be stable. In cylindrical coordinates,

$$\nabla^2 S(r) = \frac{1}{r}\frac{d}{dr}\left(r\frac{dS}{dr}\right) = S''(r) + \frac{1}{r}S'(r) . \qquad (47)$$

Substituting Eq. (47) into Eq. (45) gives the spatial equation

$$S'' + \frac{1}{r}S' + \frac{1}{\alpha\tau}S = 0 \qquad (48)$$

and substituting Eq. (42) into Eqs. (39) and (40) gives the boundary conditions

$$S(a) = 0 \qquad (49)$$

$$S'(ma) = 0 . \qquad (50)$$

Eqs. (48)–(50) can be normalized by writing

$$S(r) = y(x) \qquad (51)$$

where

$$x = \frac{r}{a} \qquad (52)$$

and defining the positive real number μ such that

$$\mu^2 = \frac{a^2}{\tau\alpha} . \qquad (53)$$

The results are

$$y'' + \frac{1}{x}y' + \mu^2 y = 0 \qquad (54)$$

$$y(1) = 0 \qquad (55)$$

$$y'(m) = 0 . \qquad (56)$$

For a given m, Eqs. (54)–(56) constitute a regular Sturm–Liouville problem [6, p. 334]. The problem has the trivial solution $y(x) = 0$, and nontrivial solutions for an infinite number of discrete values of μ, known as the eigenvalues. Let the nth eigenvalue (in ascending order) be μ_n and let the corresponding solution, called the eigenfunction belonging to μ_n, be $y_n(x)$. Each eigenfunction is determined up to an arbitrary scale factor.

6.2 Significance of Eigenvalues

From Eq. (53),

$$\tau = \frac{a^2}{\mu^2\alpha} . \qquad (57)$$

So each eigenvalue μ_n corresponds to a time constant τ_n. Substituting Eqs. (46), (51), and (52) into Eq. (42) and using a subscript n for every quantity depending on the eigenvalue μ_n, we obtain

$$\theta_n = A_n y_n\left(\frac{r}{a}\right) e^{-t/\tau_n} \qquad (58)$$

where

$$\tau_n = \frac{a^2}{\mu_n^2\alpha} . \qquad (59)$$

This is a product solution to the heat equation [Eq. (38)] satisfying the boundary conditions [Eqs. (39) and (40)]. Because the heat equation is linear and the boundary conditions are homogeneous, the equation and boundary conditions are also satisfied by any linear combination of solutions of the form of Eq. (58). So a general solution of Eqs. (38)–(40) is

$$\theta = \sum_{n=1}^{\infty} A_n y_n\left(\frac{r}{a}\right) e^{-t/\tau_n} . \qquad (60)$$

Putting $t = 0$, applying the initial condition (41), and using Eq. (52), we obtain

$$\sum_{n=1}^{\infty} A_n y_n(x) = \theta_0, \qquad 1 < x < m . \qquad (61)$$

The coefficients A_n may now be found in the usual manner by exploiting the orthogonality of the eigenfunctions [7, pp. 218–228]. Eq. (60) will then be the complete solution of Eqs. (38)–(41).

The coefficients need not be investigated further. For present purposes it suffices to note that every term or "mode" in the solution (60) has its own time constant which, according to Eq. (59), is proportional to the inverse square of the eigenvalue. The longest time constant is τ_1, corresponding to the smallest eigenvalue μ_1, and is given by

$$\tau_1 = \frac{a^2}{\mu_1^2\alpha} . \qquad (62)$$

Hence the mode associated with τ_1 (the fundamental mode) has the slowest decay and becomes dominant as t increases, that is, as equilibrium is approached.

Although we have considered the solution to the heat equation in the case of a *step* compression [see before Eq. (41)], the conclusion that the fundamental mode is dominant under near-equilibrium conditions leads to a justification for considering only the fundamental mode when analyzing sinusoidal compressions of any frequency. By causality, the rate of heat transfer at any instant is not affected by subsequent variations in applied pressure and is the same as it would be if the pressure were subsequently held constant, as it is in the case of a step compression. Thus for any applied pressure function, the pattern of heat flow at any instant can be reproduced by a step compression with the appropriate initial temperature field at that instant. In the complete solution of the heat equation, the initial temperature field is invoked only in the final step to determine the coefficients of the modes. It does not affect the eigenfunctions or time constants. Therefore as far as the instantaneous heat flux is concerned, the eigenfunctions and time constants that apply when the air is subject to a step compression also apply in other cases, including the case of sinusoidal compression. At low frequencies, for which the temperature field has ample time to "equalize" during a single cycle of compression, the departure from thermal equilibrium is small, so that the fundamental mode is dominant. At higher frequencies the higher order modes become significant. This affects the rate at which the system approaches adiabatic behavior as frequency increases, but does not alter the fact that the limiting high-frequency behavior is adiabatic. Moreover, it will be seen in Section 6.5 that the higher order modes allow only a small fraction of the air volume to exchange heat with the fiber, so that they cannot, by themselves, cause gross departure from adiabatic behavior. Hence at all frequencies only a small error is incurred by assuming that all heat conduction is due to the fundamental mode.

6.3 Computation of Eigenvalues and Eigenfunctions

The Sturm–Liouville problem comprises Eqs. (54)–(56). Because the eigenfunctions can be scaled arbitrarily, let us normalize them by introducing a fourth equation,

$$y'(1) = 1 . \tag{63}$$

The four equations can then be conveniently regrouped as an initial-value problem (IVP),

$$y'' + \frac{1}{x}y' + \mu^2 y = 0$$

$$y(1) = 0 \tag{64}$$

$$y'(1) = 1$$

with a remote boundary condition

$$y'(m) = 0 . \tag{65}$$

The IVP has a unique solution for each value of μ. The eigenvalues are the values of μ for which the IVP solution satisfies Eq. (65).

For $\mu = 0$, Eqs. (64) have the exact solution

$$y = \ln x . \tag{66}$$

This is the limiting solution of the IVP as $\mu \to 0$. For $\mu > 0$, let $s(n, \mu)$ be the abscissa of the nth stationary point of $y(x)$ as x increases from $x = 1$. Then μ_n is the value of μ for which

$$s(n, \mu) = m . \tag{67}$$

A numerical algorithm for computing the eigenvalues and eigenfunctions is described in [8, sec. 8.2.4]. The results for the first three eigenvalues, together with the limiting function $y = \ln x$, are shown in Fig. 3. It can be seen at a glance that each eigenfunction satisfies the boundary conditions in Eqs. (64) and (65) and has the required number of extrema on $[1, m]$.

6.4 Comparison with Chase (1974) and Leach (1989)

Both Chase [2] and Leach [1] solve the heat equation by the separation of variables, with a transformation of the radial equation [Eq. (54)] to Bessel's equation. Both authors give a complete series solution in the form of Eq. (60), except that they use the notation $V_0(\mu_n x)$ in place of $y_n(x)$ and use a instead of α for the thermal diffusivity.

For $m = 20$ (the only numerical example that he considers), Leach gives the eigenvalue $\mu_1 = 0.232$, which matches μ_2 in Fig. 3 to three decimal places. So Leach has found μ_2 instead of μ_1, leading to the time constant τ_2 instead of τ_1. If we use $a = 5$ μm and $\alpha = 1.87 \times 10^{-5}$ m^{2}s^{-1}, as quoted by Leach, and take μ_2 from Fig. 3, we obtain $\tau_2 = 24.8$ μs. This corresponds to a transition frequency of 6.4 kHz, which is Leach's result. If Leach had used μ_1 from Fig. 3, he would have obtained a transition frequency of 258 Hz, which is much closer to my result of 289 Hz.

Chase's Eq. (7), which corresponds to my Eq. (62), has the outer radius ma (which Chase calls $R_2 = mR_1$) in place of the inner radius a (which he calls R_1). Leach's Eq. (36) corrects Chase's Eq. (7). However, this substitution does not fully account for Chase's results; further details are given in [8, sec. 8.2.5].

6.5 Why Higher Order Modes Are Neglected

We can now finish the argument that concludes Section 6.2. In Fig. 3 recall that x is proportional to the radius and y is proportional to the excess temperature of the associated mode. Thus every maximum or minimum in an eigenfunction represents a *modal heatshed*. For the nth mode, only the air inside the radius corresponding to the first maximum of the eigenfunction y_n can exchange heat with the fiber. For $m = 20$, Fig. 3 shows that the radius of the first maximum of y_2 is less than 30% of the outer radius of the heat tube, so that the fraction of the air volume that can exchange heat with the fiber in the second mode, which fraction we shall call f_2, is less than 9%. In the third and higher modes, the fraction f_2 is smaller. Thus the higher order modes seem unimportant.

To see whether the same conclusion holds for all realistic values of m, we need to compute f_2 for a range of values of the filling factor f. This exercise has been done in [8, sec. 8.2.6]. It was found that f_2 rises to 11% for $f = 1\%$ and to 13% for $f = 2\%$, and theoretically approaches a limit of 1/3 as $f \rightarrow 100\%$. So for realistic filling factors, f_2 remains small. When we remember that air is only 40% more compliant under isothermal compression than under adiabatic compression, the significance of these small values of f_2 is further diminished.

7 REFINING THE ANALYTICAL APPROXIMATION

An estimate of the first eigenvalue μ_1 can be found from the analytical approximation to the time constant. Writing τ_1 for τ_{fp} in Eq. (31) and substituting from Eq. (62) gives

$$\mu_1 \approx \left[\frac{1}{2}(m^2 - m^\zeta) \ln\left(\frac{m+1}{2}\right)\right]^{-1/2} . \quad (68)$$

For large m we can estimate μ_1 by setting $\zeta = 0$ [see Eqs. (30)–(32) and the subsequent discussion]. Let this "rough" estimate of μ_1 be called μ_a, and let the "refined" estimate based on a general value of ζ be called μ_b. Then we have

$$\mu_a = \left[\frac{1}{2}(m^2 - 1) \ln\left(\frac{m+1}{2}\right)\right]^{-1/2} \quad (69)$$

and

$$\mu_b = \left[\frac{1}{2}(m^2 - m^\zeta) \ln\left(\frac{m+1}{2}\right)\right]^{-1/2} . \quad (70)$$

The numerical computations of μ_1 can be checked by comparing μ_1 with μ_a for large m. Then we can adjust the parameter ζ to obtain the best possible agreement between μ_b and μ_1 for the full range of m. The necessary comparisons and adjustments are made easier if we tabulate the percentage errors in the two analytical estimates of μ_1, rather than the estimates themselves. The algorithm for checking a single trial value of ζ is described in [8, p. 137].

By trial and error it was found that $\zeta = 0.37$ is about optimal; in the printout for $\zeta = 0.37$, shown in Table 1, the percentage errors in μ_a and μ_b appear as ermua and ermub. (The printout contains some additional diagnostics. The eigenvalue μ_1 was computed using two step sizes, the second being half the first. The number of steps between $x = 1$ and $x = m$ for the smaller step size is tabulated as steps. The difference between the two results, in parts per million, appears in the printout as errmu. The worst discrepancy is seen to be about 25 ppm, indicating that step size is not a significant source of error.)

For the whole range of filling factors, the rough estimate μ_a differs from μ_1 by no more than 10%. As expected, the error is much smaller for large m, suggesting that the numerical analysis is free from gross errors.

For $\zeta = 0.37$, the "refined" analytical formula underestimates μ_1 by a margin of less than 0.9% for all m. By Eq. (62), $\tau_1 \propto \mu_1^{-2}$, so that the percentage error in the estimate of τ_1 will be -2 times that in the estimate of μ_1. Hence Eq. (31) overestimates τ_1 by no more than 1.8%. Now all the analysis in this chapter has neglected the thermal resistance of the glass. Recalling that the thermal resistivity of the air is about 40 times that of the glass, and supposing that most of the air-to-glass thermal resistance is caused by the regions close to the air–glass surface, inclusion of the thermal resistance of the glass would increase the air-to-glass resistance, and hence the thermal time constant, by (very roughly) one part in 40, or 2.5%. Hence overestimating τ_1 tends to compensate for the neglect of the thermal resistance of the glass. For this reason it seems prudent to accept the consistent underestimation of μ_1 shown in the rightmost column of Table 1.

Putting $\zeta = 0.37$ in Eq. (32) gives

$$\tau_{fp} \approx \frac{d^2}{8\alpha}(m^2 - m^{0.37}) \ln\left(\frac{m+1}{2}\right) . \quad (71)$$

At a temperature of 20°C and a pressure of 1 atm, the

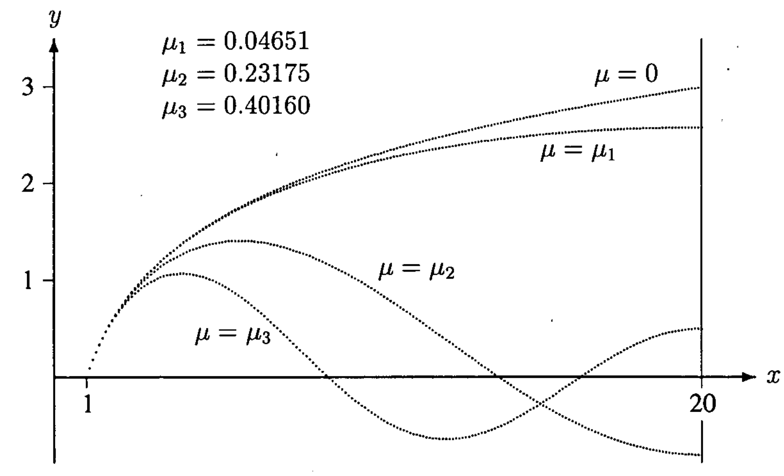

Fig. 3. Radial eigenfunctions for $m = 20$, and limiting function.

thermal diffusivity α is about 2.121×10^{-5} m^{2}s^{-1} (estimated from the formulas in [9]). If we put $d = 10$ μm and $f = 0.25\%$ (whence $m = 20$), Eq. (71) gives $\tau_{fp} \approx$ 550 μs, which corresponds to the transition frequency of 289 Hz quoted in the Introduction.

Having chosen ζ, we can estimate the ultimate accuracy of Eq. (71). For filling factors up to 8%, Eq. (70) underestimates μ_1 by 0.34–0.83%; hence Eq. (71) overestimates τ_1 by 0.68–1.66%. Suppose that, because we have neglected the thermal resistance of the glass, τ_1 underestimates τ_{fp} by 2.5%. Then Eq. (71) underestimates τ_{fp} by 0.84–1.82%. We may therefore reasonably expect Eq. (71) to be correct to within 2%, provided that the filling factor is small enough to justify the cylindrical-heat-tube assumption. The neglect of higher order (that is, shorter) time constants has a qualitatively similar effect to overestimating a single time constant, and therefore tends to compensate for an underestimation of τ_{fp}. So the calculated value of τ_{fp} will still be within 2% of the effective value provided that the departure from thermal equilibrium is not too great.

The requirements of a small filling factor and a small departure from thermal equilibrium have not been quantified. Concerning the filling factors likely to be encountered in practice, Chase's Fig. 1 [2, p. 299] indicates that common fiberglass with a natural density of 6 kg m^{-3} has a "practical maximum compressed density" of 70 kg m^{-3}. The natural and maximum densities correspond to filling factors of 0.25% and 2.9%, respectively. According to Bradbury [10, p. 163], the filling factor is at most 5%.

8 THERMAL TIME CONSTANTS AND THE ACOUSTIC CIRCUIT

The acoustic compliance of a volume V under adiabatic compression is [1]

$$C_a = \frac{V}{\rho_0 c^2}.$$

If the volume in question is fiber filled, V must be replaced by the air volume $V(1 - f)$, so that the acoustic compliance becomes

$$C_a = \frac{V(1-f)}{\rho_0 c^2}. \tag{72}$$

This assumes that the compressions are still adiabatic, which in practice means that the frequency is too high for significant heat conduction between the air and the fibers. If, on the contrary, the frequency is so low that the air and the fibers can be assumed to be in thermal equilibrium, the compliance is higher. The increase can be modeled by an additional compliance C_{th} (which we may call the thermal relaxation compliance) in parallel with C_a. We require C_{th} to be effective only at low frequencies. The obvious way to achieve this is to place a resistance R_{th} (the thermal relaxation resistance) in series with C_{th}. Thus by a simple intuitive argument we infer that the form of the equivalent circuit is as shown in Fig. 4.

This form was derived by Leach [1, pp. 587–589] by assuming that in response to a step change in pressure, the apparent volume of the box begins at the adiabatic value and exponentially approaches the thermal-equilibrium value. A Laplace transform then leads to the equivalent circuit. (Because Leach assumes an isobaric heat transfer, his time constant τ_f is identical to my τ_p; his subscript f apparently stands for constant force, which implies constant pressure, whereas my subscript f stands for constant fiber temperature.) I have derived the same circuit by a third method, involving simultaneous differential equations, in [8, sec. 7.2].

To find the component values in Fig. 4, first note that the excess pressure p appears across C_a. If p is constant, no flux (volume velocity) flows in C_a, so that all of the external flux u_{in} flows in R_{th} and C_{th}, that is, when the air expands, the volume of air flowing out of the volume V is supplied by "discharging" C_{th}, so that the pressure

Table 1. Computer output showing errors in analytical approximations to μ_1 for various filling factors, for $\zeta = 0.37$.*

f (%)	m	mu1	steps	errmu	ermua	ermub
0.016	80.00	0.009224	7901	10.20	−0.38	−0.34
0.031	56.57	0.013698	5557	11.29	−0.42	−0.36
0.062	40.00	0.020448	3901	12.57	−0.48	−0.39
0.125	28.28	0.030717	2729	13.95	−0.58	−0.43
0.250	20.00	0.046509	1900	15.70	−0.73	−0.47
0.500	14.14	0.071135	1315	17.81	−0.95	−0.53
1.000	10.00	0.110270	900	20.07	−1.28	−0.60
2.000	7.07	0.174116	608	22.85	−1.76	−0.68
4.000	5.00	0.282361	401	25.12	−2.46	−0.76
8.000	3.54	0.477237	254	23.92	−3.43	−0.83
19.237	2.28	1.034493	201	0.00	−5.14	−0.84
37.180	1.64	2.216110	200	−12.26	−6.83	−0.74
57.392	1.32	4.636943	200	−15.63	−8.12	−0.59
74.316	1.16	9.524470	201	−15.72	−8.96	−0.46
85.734	1.08	19.329821	201	−15.39	−9.44	−0.38
92.456	1.04	38.958271	200	−17.63	−9.70	−0.33
96.117	1.02	78.224464	201	−12.87	−9.83	−0.30

* The sixth and seventh columns show percentage errors in the "rough" and "refined" analytical approximations.

in C_{th} is proportional to the excess volume, with a sign reversal. But at constant pressure, specific volume is proportional to temperature, so that the excess volume is proportional to the excess temperature θ. Therefore the pressure in C_{th} is proportional to $-\theta$, so that the thermal time constant is just the time constant of R_{th} and C_{th}, that is,

$$\tau_p = R_{th}C_{th} . \tag{73}$$

At constant volume (or constant density), the pressure and temperature of the air are proportional, so that the excess pressure appearing across C_a is proportional to the excess air temperature θ. Therefore the thermal time constant is the time constant applicable to the pressure across C_a. Now at constant volume there is no external flux (that is, $u_{in} = 0$), so that the equivalent circuit is isolated and its time constant is simply that of the series *CRC* circuit, that is,

$$\tau_v = R_{th}(C_a \| C_{th}) . \tag{74}$$

Substituting Eq. (13) into Eq. (73) gives

$$R_{th}C_{th} = \frac{\beta\tau_{fp}}{\gamma + \beta} . \tag{75}$$

Substituting Eq. (14) into Eq. (74) gives

$$R_{th}(C_a \| C_{th}) = \frac{\beta\tau_{fp}}{\gamma(1 + \beta)} . \tag{76}$$

Thus we have two equations in the two unknowns C_{th} and R_{th}. The solutions are

$$C_{th} = \frac{(\gamma - 1)\beta C_a}{\gamma + \beta} \tag{77}$$

and

$$R_{th} = \frac{\tau_{fp}}{(\gamma - 1)C_a} \tag{78}$$

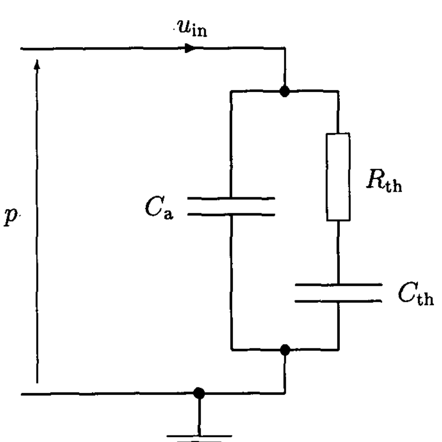

Fig. 4. Equivalent circuit for compressibility of air in fiber-filled volume.

where C_a is given by Eq. (72), β by Eq. (12), and τ_{fp} by Eq. (71).

Eqs. (72), (77), and (78) can be reconciled with the corresponding results of Leach [1, p. 589] using the following substitutions, in which Leach's notations appear on the left:

$$V_B(1 - V_f/V_B) \rightarrow V(1 - f)$$

$$C_{AB1} \rightarrow C_a$$

$$C_{AB2} \rightarrow C_{th}$$

$$V_B/V_f \rightarrow 1/f$$

$$R_{AB1} \rightarrow R_{th}$$

$$\tau_f \rightarrow \tau_p = \frac{\beta\tau_{fp}}{\gamma + \beta} .$$

Leach's use of τ_{fp} as an estimate of τ_p is corrected by the last substitution, which quotes Eq. (13). With these six substitutions, Leach's Eq. (17) becomes my Eq. (72). His Eq. (18) yields my Eq. (77) with the aid of my Eq. (12), and his Eq. (19) yields my Eq. (78) with the aid of my Eq. (77).

9 DISCUSSION

For a fiber diameter of 10 μm and a filling factor of 0.25%, Leach [1] concludes that the air and the fibers may be assumed to be in thermal equilibrium up to a transition frequency of about 6.4 kHz, whereas the present paper gives a transition frequency of about 289 Hz. It must be conceded, however, that a gross error in modeling the thermal behavior and the dynamic compressibility of the air does not necessarily translate into a gross error in predicting the output of the loudspeaker. Whether the transition frequency is 6.4 kHz or 289 Hz, we can assume thermal equilibrium for calculating the bass rolloff characteristic. At frequencies just above rolloff, the motion of the diaphragm is mass limited, so that the dynamic compressibility of the air is less significant. At higher frequencies, the diaphragm is heavily influenced by unwanted internal resonances in the enclosure, and a major purpose of the filling material is to dampen such resonances. Because thermal relaxation is a mechanism of damping, the thermal time constant must have some effect on the frequency-response irregularities caused by resonances. The examples considered in [8, chap. 10, esp. figs. 10.13 and 10.22] suggest that the viscosity of the air is the dominant mechanism of damping resonances, so that the assumption of thermal equilibrium causes only a small error in the computed frequency response. But it remains conceivable that certain combinations of fiber diameters, filling factors, and operating frequencies could make the departure from thermal equilibrium more significant, justifying the use of the more precise model of dynamic compressibility shown in Fig. 4. As this model is quite simple, its use requires little justification.

10 FURTHER WORK

Lest the elegance of Eq. (71) obscure its limitations, it should be noted that the "accuracy" of 2% accounts only for the difference between analytical and numerical computations and (very roughly) for the effects of the thermal resistivity of the fibers. The assumption of cylindrical heat tubes requires a small filling factor, but "small" has not been quantified. The effect of neglecting higher order modes has not been quantified, except to say that it leads to the correct limiting behavior at high and low frequencies and that the volume fraction affected by higher order modes is small. No attempt has been made to account for fibers with nonuniform diameters or irregular geometries.

If the error in Eq. (71) were to be determined by experiment, the error in measuring τ_{fp} would need to be substantially smaller than the error in the formula. Such accuracy seems unlikely in view of the variable properties of fiberglass filling material—for example, handling the material causes nonuniform and unpredictable changes in its packing density. Probably the best we can hope for is an approximate measurement of τ_{fp}, offering further confirmation that the computations in this paper are of the right order.

A direct measurement of any of the five time constants defined in this paper seems impossible because the conditions under which the time constants are defined could not be imposed. Moreover, the measurement would be imprecise because of the time constant of the thermometer and the nonzero time taken to set up the initial conditions. Therefore any measurement of τ_{fp} would need to be indirect, involving the measurement of some acoustic property of the air–fiber medium that depends on τ_{fp}. Damping is apparently not a suitable property for this purpose because, as noted, it is more heavily influenced by viscous effects than by thermal effects, and the modeling of viscous effects brings its own errors. The speed of sound c suggests itself as an indirect measure of τ_{fp} because it depends on compressibility, which in turn varies with frequency in a manner dependent on τ_{fp}. But c also depends on density, and the effective density depends on whether the fibers move with the air, which again is complicated by the viscosity question. Whatever acoustic property is measured, it seems that a viable experiment requires a system in which viscosity and heat conduction exert their effects over different frequency ranges, so that their effects can be distinguished.

While it may be possible to devise such an experiment, the author has not yet done so, and believes that the exercise requires a separate paper. In the meantime, the agreement between the two methods used to obtain the results in this paper gives some cause for confidence.

11 CONCLUSION

If viscosity, rather than heat conduction, is the dominant mechanism of damping in loudspeaker enclosures, an improved model of heat conduction is only a modest contribution to the art of modeling loudspeakers. Nevertheless, as the following summary shows, the model recommends itself by its simplicity.

The compressibility of air in a fiber-filled volume can be modeled as the well-known adiabatic compliance C_a in parallel with the series combination of a thermal relaxation compliance C_{th} and a thermal relaxation resistance R_{th}. The form of the equivalent circuit confirms the earlier result of Leach [1]. Formulas have been derived for computing C_{th} and R_{th} from the properties of the air and the filling material. These formulas also confirm those given by Leach, except that the time constant used in the calculation of R_{th} has been more accurately expressed in terms of τ_{fp}, which is the thermal time constant at constant fiber temperature and constant pressure.

Moreover, τ_{fp} may be estimated from the formula

$$\tau_{fp} \approx \frac{d^2}{8\alpha}(m^2 - m^{0.37}) \ln\left(\frac{m+1}{2}\right) \tag{79}$$

where d is the fiber diameter, α is the thermal diffusivity of air, f (not in the formula) is the filling factor, and $m = f^{-1/2}$. This simple formula agrees with a more elaborate numerical simulation to within 2%.

The numerical results given by Eq. (79) differ greatly from those given by Leach [1] and Chase [2]. The disagreement with Leach is explained by his choice of eigenvalue in the radial Sturm–Liouville problem. The disagreement with Chase is partly explained by an incorrect substitution, which was also noted by Leach. But the main argument in support of Eq. (79) is that it comes from a synthesis of two independent methods, whereas Leach and Chase relied on a single method. The explicit formula for τ_{fp} is also more convenient to use than previously published results.

12 ACKNOWLEDGMENT

This work was undertaken while the author was a Ph.D. candidate in the Department of Electrical and Computer Engineering, University of Queensland, under the supervision of Dr. L. V. Skattebol. The results presented in this paper are essentially contained in the author's thesis [8]. However, the thesis treats thermal time constants in the context of a finite-difference equivalent-circuit model of the distributed inertance and compliance of a fiber-filled enclosure, whereas this paper sacrifices some generality and rigor in favor of a shorter and more accessible treatment.

13 REFERENCES

[1] W. M. Leach, Jr., "Electroacoustic-Analogous Circuit Models for Filled Enclosures," *J. Audio Eng. Soc.*, vol. 37, pp. 586–592 (1989 July/Aug.).

[2] L. M. Chase, "The Thermo-Acoustic Properties of Fibrous Material," *IEEE Trans. Acoust., Speech, Signal Process.*, vol. ASSP-22, pp. 297–300 (1974 Aug.).

[3] D. E. Gray (ed.), *American Institute of Physics Handbook*, 3rd ed. (McGraw-Hill, New York, 1972).

[4] K. Kadoya, N. Matsunaga, and A. Nagashima,

"Viscosity and Thermal Conductivity of Dry Air in the Gaseous Phase," *J. Phys. Chem. Ref. Data*, vol. 14, pp. 947–970 (1985).

[5] D. R. Lide (ed.), *CRC Handbook of Chemistry and Physics*, 74th ed. (CRC Press, Boca Raton, FL, 1993).

[6] P. V. O'Neil, *Advanced Engineering Mathematics* (Wadsworth, Belmont, CA, 1983).

[7] E. Kreyszig, *Advanced Engineering Mathematics*, 6th ed. (Wiley, New York, 1988).

[8] G. R. Putland, "Modeling of Horns and Enclosures for Loudspeakers," Ph.D. dissertation, University of Queensland, Brisbane, Australia (1996). http://www.users.bigpond.com/putland/phd/index.html

[9] G. R. Putland, "Acoustical Properties of Air versus Temperature and Pressure," *J. Audio Eng. Soc. (Engineering Reports)*, vol. 42, pp. 927–933 (1994 Nov.).

[10] L. J. S. Bradbury, "The Use of Fibrous Materials in Loudspeaker Enclosures," *J. Audio Eng. Soc.*, vol. 24, pp. 162–170 (1976 Apr.), reprinted in *Loudspeakers: An Anthology*, 2nd ed. (Audio Engineering Society, New York, 1980), pp. 404–412.

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Dr. Putland has also held two patents (not related to audio) and has studied some units of New Testament Greek and Modern Greek.

By the time this paper is published, Dr. Putland's contract will have expired, his Greek studies will have resumed, and his old department will have merged with another to form the new Department of Computer Science and Electrical Engineering.